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# Transverse Shear Deformation in Exact Buckling and Vibration of Composite Plate Assemblies

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## Introduction

THE program VICONOPT[1] performs buckling and vibration analysis and optimum design of any prismatic assembly of composite plates. The analysis assumes that the response in the longitudinal direction is a summation of sinusoidal responses and that individual plates have stiffness properties resulting from balanced symmetric laminates. It uses stiffness matrices derived analytically from the exact solution of the uncoupled in-plane and out-of-plane fourth-order differential equations of classical plate theory. The introduction of transverse shear deformation makes the out-of-plane equation sixth order, necessitating a numerical solution. In this Note the numerical solution is developed in a general way to include the coupled case that has order $N = 10$ when transverse shear is included and order $N = 8$ otherwise, so allowing the exact treatment of any laminate with an arbitrarily located reference surface.

The usual numerical approach involves writing the coupled plate equilibrium equations in terms of displacements and determining the characteristic roots from the associated determinant. This approach leads to complicated expressions that are impractical to implement for the most general cases. Alternatively, Cohen[2] wrote the differential equations as a first-order system with the displacements and the associated forces as unknowns. The resulting matrix is much simpler and can be arranged so that the characteristic root appears only on the diagonal, so that a standard linear eigenvalue solver can be used. The eigenvectors give the relationships between all of the forces and displacements.

Because exact stiffness expressions are used, plates need not be subdivided to obtain accuracy. Nodes are required only at junctions of two or more plates. At such junctions, continuity of rotation requires that the shear angle in the plane of the edge forming a junction must be zero unless all plates are coplanar. By using the shear angle as the additional unknown over classical plate theory, as in Cohen,[2] the continuity condition is satisfied by deleting the rows and columns of the plate stiffness matrix corresponding to the shear angle. This results in a stiffness matrix with the same unknowns as classical plate theory, and the assembly of the global stiffness matrix from individual plate stiffnesses is unchanged.

This Note summarizes the development of the theory for which Ref. 3 gives a complete description, including all necessary equations for its direct implementation and typical results obtained from its application.

## Governing Plate Equations

It is assumed that the plate has fully populated $\underline{A}$, $\underline{B}$, and $\underline{D}$ stiffness matrices with its reference surface arbitrarily located in the $x$-$y$ plane as shown in Fig. 1. First-order shear deformation theory, where the in-plane displacements $u$ and $v$ vary linearly with $z$, is assumed. The plate is loaded by uniform in-plane stress resultants (see Fig. 1a) acting at a distance $z_c$ from the reference surface in the centroidal plane of the cross section treated as a wide beam. Figure 1b shows the additional forces and moments that occur during buckling or the amplitudes of such forces during vibration at a frequency $\omega$. Deflection in all three directions is resisted by Winkler elastic foun-

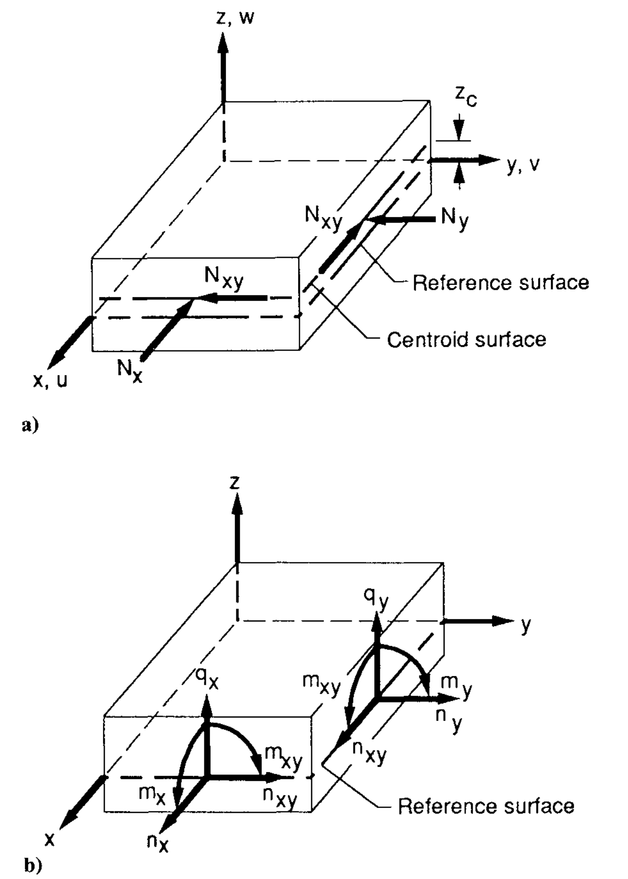


Fig. 1 Positive directions of forces and moments per unit width acting on a plate element: a) prebuckling in-plane loads ($N_x$, $N_y$, $N_{xy}$) and b) buckling in-plane forces ($n_x$, $n_y$, $n_{xy}$), moments ($m_x$, $m_y$, $m_{xy}$), and transverse shearing forces ($q_x$, $q_y$).

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dations of stiffness $K_x$, $K_y$, and $K_z$. The equilibrium equations are

$$n_{x,x} + n_{xy,y} - N_x u_{0,xx} - K_x u + 4\pi^2\omega^2(m_0 u - m_1\psi_x) = 0$$

$$n_{xy,x} + n_{y,y} - N_x v_{0,xx} - K_y v + 4\pi^2\omega^2(m_0 v - m_1\psi_y) = 0$$

$$q_{x,x} + q_{y,y} - K_z w - N_x w_{,xx} - N_y w_{,yy} - 2N_{xy} w_{,xy} + 4\pi^2\omega^2 m_0 w = 0 \tag{1}$$

$$m_{xy,x} + m_{y,y} - q_y - N_x z_c v_{0,xx} + 4\pi^2\omega^2(m_1 v - m_2\psi_y) = 0$$

$$m_{x,x} + m_{xy,y} - q_x - N_x z_c u_{0,xx} + 4\pi^2\omega^2(m_1 u - m_2\psi_x) = 0$$

where a comma indicates differentiation with respect to the variables that follow, $m_j$ is the $j$th moment of mass about the reference surface, and $\psi_x$ and $\psi_y$ are rotations of the normals to the reference surface about the $y$ and $x$ axes, respectively. These equations are generalized from those of Ref. 4 to allow for the applied load $N_x$ to act at the centroidal surface rather than the reference surface. As in Ref. 4, it is assumed that the only force to affect the in-plane equilibrium equations is $N_x$. The in-plane displacements in the centroidal plane are given by

$$u_0 = u - z_c\psi_x ; \qquad v_0 = v - z_c\psi_y \tag{2}$$

## Stiffness Matrix

The objective of the analysis is to derive a stiffness matrix that relates the amplitudes of the displacements and forces along the longitudinal edges $y = \pm b/2$. The desired displacement and force variables are

$$\underline{z}(y) = \begin{bmatrix} \underline{d} \\ \underline{f} \end{bmatrix} = \begin{bmatrix} (\,iu \quad v \quad w \quad \psi_y \quad i\gamma_x)^T \\ (in_{xy} \quad n_y \quad Q_y \quad m_y \quad im_{xy})^T \end{bmatrix} \tag{3}$$

where the use of $\gamma_x = w_{,x} - \psi_x$ as a fundamental displacement variable rather than the rotation $\psi_x$ preserves continuity of rotations at plate junctions. The introduction of $i = \sqrt{-1}$ results in real plate stiffnesses for orthotropic plates without shear loading. $Q_y$, the transverse shearing edge force in the $z$ direction, is made normal to the reference surface of the undeflected plate by replacing $q_y$ with the Kirchhoff value

$$q_y = Q_y - m_{xy,x} + N_y w_{,y} + N_{xy} w_{,x} \tag{4}$$

The problem is changed to an ordinary differential equation in $y$ by assuming a sinusoidal variation in the $x$ direction with half-wavelength $\lambda$, so that the displacements and forces throughout the plate are given by

$$\underline{Z}(x,y) = \exp(i\pi x/\lambda)\ \underline{z}(y) \tag{5}$$

The next step is to express all unknowns in terms of $\underline{z}$. The stress resultant strain relationships given in terms of the $\underline{A}$, $\underline{B}$, and $\underline{D}$ stiffness matrices are partially inverted to give needed quantities in terms of the fundamental variables or terms derivable from them without any $y$ derivatives.

$$(n_x\ \epsilon_y\ \epsilon_{xy}\ m_x\ \kappa_y\ \kappa_{xy})^T = \underline{H}_c(\epsilon_x\ n_y\ n_{xy}\ \kappa_x\ m_y\ m_{xy})^T$$
$$(q_x\ \gamma_y)^T = \underline{H}_s(\gamma_x\ q_y)^T \tag{6}$$

where $\epsilon_x = u_{,x}$ and $\kappa_x = -\psi_{x,x}$. $\underline{H}_c$ is calculated from the $\underline{A}$, $\underline{B}$, and $\underline{D}$ stiffness matrices of laminate theory, whereas $\underline{H}_s$ is determined from Cohen.[5] Equation (4) is also written without any $y$ derivatives using the relation

$$w_{,y} = \gamma_y - \psi_y \tag{7}$$

with $\gamma_y$ given by Eq. (6). The use of Eqs. (4), (6), and (7) allows the strain displacement equations

$$\epsilon_{xy} = u_{,y} + v_{,x}, \qquad \epsilon_y = v_{,y}, \qquad \gamma_y = w_{,y} - \psi_y$$
$$\kappa_y = -\psi_{y,y}, \qquad \kappa_{xy} = \gamma_{x,y}, \quad \gamma_{y,x} - 2\psi_{y,x} \tag{8}$$

and the equilibrium equations to be written in terms of the elements of $\underline{z}$ as

$$\underline{z}' = \underline{P}\,\underline{z} \tag{9}$$

where a prime denotes differentiation with respect to $y$. The elements of $\underline{z}$ are assumed to be given by

$$z_j = c_j \exp(i\beta y/b) \tag{10}$$

where $\beta$ is one of the $N$ characteristic roots of the differential equation. Substituting Eq. (10) into Eq. (9) gives

$$(\underline{R} - \beta\underline{I})\ \underline{c} = \underline{0} \tag{11}$$

where $\underline{I}$ is the identity matrix and $\underline{c}$ is the vector formed from the $c_j$ of Eq. (10). Thus the values of $\beta$ are the eigenvalues of $\underline{R}$, which is not symmetric but which consists of pure real and pure imaginary elements and can be made real[3] by multiplying appropriate rows and columns by $\pm i$. If $\underline{C}$ is a matrix whose columns are the eigenvectors $\underline{c}$, its upper half $\underline{a}$ will consist of displacements and its lower half $\underline{b}$ of forces.

Denoting quantities evaluated at $y = -b/2$ and $y = b/2$ by superscripts 1 and 2, respectively, the amplitudes of the displacements and forces at the two edges of the plate may be written as

$$\begin{bmatrix} d_j^1 \\ d_j^2 \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} a_{jk} r_k \exp(-i\beta_k/2) \\ a_{jk} r_k \exp(i\beta_k/2) \end{bmatrix}$$
$$\begin{bmatrix} f_j^1 \\ f_j^2 \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} b_{jk} r_k \exp(-i\beta_k/2) \\ b_{jk} r_k \exp(i\beta_k/2) \end{bmatrix} \tag{12}$$

where the $r_k$ are constants to be determined from the edge values. Equation (12) can be written in matrix form as

$$\begin{bmatrix} \underline{d}^1 \\ \underline{d}^2 \end{bmatrix} = \underline{E}\,\underline{r}\,, \qquad \begin{bmatrix} \underline{f}^1 \\ \underline{f}^2 \end{bmatrix} = \underline{F}\,\underline{r} \tag{13}$$

or, eliminating $\underline{r}$, as

$$\begin{bmatrix} \underline{f}^1 \\ \underline{f}^2 \end{bmatrix} = \underline{K} \begin{bmatrix} \underline{d}^1 \\ \underline{d}^2 \end{bmatrix} \tag{14}$$

where $\underline{K}$ is the desired stiffness matrix $\underline{FE}^{-1}$.

The rows and columns of $\underline{K}$ corresponding to $\gamma_x$ are deleted to obtain a stiffness matrix of the same size and involving the same unknowns as for the classical case. (An alternative procedure yielding a lower bound solution is to reduce the stiffness matrix to the same unknowns by setting $m_{xy}$ to zero.) As for the classical case, $\underline{K}$ is real and symmetric for orthotropic plates without shear loading and Hermitian otherwise. It can be used in the assembly of the global stiffness matrix without any further alteration in the coding.

Obtaining accurate numerical stiffnesses by routine reduction of the equations for $\underline{K}$ can be a problem for certain ranges of parameters. Reference 3 describes special techniques used to preserve accuracy.

### Eigenvalues for Clamped Edges

The iterative analysis procedure of VICONOPT[1] requires the plate stiffnesses to be evaluated at a series of trial values of the eigenvalue [critical buckling load factor or natural frequency of vibration, and not to be confused with the eigenvalues of the $\underline{R}$ matrix of Eq. (11)] that converge to the desired result. For each trial value the analysis[4] requires not only the plate stiffnesses but also the number of eigenvalues exceeded for each individual plate assuming its edges were clamped. The number of eigenvalues exceeded is obtained by dividing the plate width $b$ by $2^n$, with $n$ chosen to obtain a plate small enough to guarantee that none of its clamped edge eigenvalues are exceeded. Using this divided plate as a substructure that is repeatedly doubled to return to the original width allows the number of eigenvalues exceeded for clamped edges to be determined. A suitable value of $n$ is determined as follows.

Reference 3 shows that every term of the $\underline{R}$ matrix is proportional to $b$, and thus the eigenvalues of $\underline{R}$ are proportional to $b$. Noting that an eigenvalue of $\underline{R}$ equal to $\pi$ corresponds to buckling or vibration with simply supported edges, choosing $n$ such that all the real eigenvalues of $\underline{R}$ are less than $2^n\pi$ gives $b/2^n$ as the width of a plate for which no buckling or vibration eigenvalues are exceeded if the edges are simply supported, and consequently none are exceeded if the edges are clamped.

### Results

Results of the shear deformation analysis applied to the buckling of sandwich plates composed of isotropic layers agree with various published results. Vibration results[3] for a composite cylinder give good agreement with the elasticity solution,[6] indicating that first-order shear deformation theory is adequate for eigenvalue problems. Further results for a typical design problem[3] showed that a stiffened panel with sandwich construction gives significant mass savings compared with solid composite construction. The analysis also indicates the importance of accounting for shear deformations when low mass core materials are used.

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# Design Sensitivity Analysis of Structural Frequency Response

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## Introduction

In many modern engineering applications it is desirable to find the effects of design parameter changes on the dynamic response of a system. To date, most research done in this area is on the eigenproblem sensitivity analysis.[1,2] The result of such work is fruitful and almost conclusive. However, there seems to be far less work being done directly on the dynamic response sensitivities, which have even more practical applications. This Note demonstrates the derivation of an improved method for calculating design sensitivities of structural frequency response.

Conventionally, the frequency response of a structure is computed by either the direct or the modal superposition formulation.[3] By differentiating the two types of frequency response equations we can also come up with two kinds of formulations, namely, the direct and the modal formulations.[4] The former is based on the direct frequency response solution and results in an exact calculation of frequency response derivatives for all cases permitted by the analysis formulation. Despite its superb accuracy, the method is accompanied by two drawbacks. The first one is its prohibitive cost of computation when a large system is involved. This is due to its requirement for inverting the system-size matrices as many times as the total number of forcing frequencies involved. The second drawback is its inability in handling modal damping, which can become a vital concern in some applications. The latter takes advantage of the modal formulation, which does not require inverting large matrices. However, it requires computing the eigenvector derivatives, which may sometimes be cost prohibitive. In addition, when there are repeated modes existing among the active modes, the method may fail to obtain the correct result.

An alternative approach is similar to the modal superposition method for the frequency response, which employs a direct modal transformation for the response sensitivities.[5] This method has many practical advantages. It handles a very general class of frequency response problems with great computational efficiency. The primary potential problem associated with this method is the accuracy issue.

This Note provides an improved method to the modal superposition method for calculating frequency response sensitivities. It combines the mode-acceleration method with the Ritz minimization technique to improve the modal approximation accuracy. A similar approach was introduced by Wang for calculating eigenvector derivatives with a very successful result.[6] In addition, an iteration scheme is used to further enhance the result. Such an approach has also been used for eigenvector derivatives.[7,8]

## Theoretical Background

#### Frequency Response

For a harmonic excitation, the structural response is expected to be also harmonic with the same frequency. The

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